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Some results on the asymptotic distribution of the zeros of orthogonal polynomials

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Abstract: Define a discrete measure that attributes masses of size $1/n$ at every zero of the polynomial of degree n, orthogonal with respect to some positive measure on the real line. The weak limit of this sequence of measures is studied by many authors. In this paper we give some results on the rate of this asymptotic behaviour, in particular the results show how the behaviour of the spectral measure, with respect to which the polynomials are orthogonal, affects this asymptotic zero behaviour.

Keywords: Orthogonal polynomials, zeros of polynomials, asymptotics.

1. Introduction

Let $(p_n(x);\ n = 0, 1, 2,\dots)$ be a sequence of polynomials, orthogonal with respect to a positive measure λ on the real line:

$$\int p_n(x)p_m(x)\,\mathrm{d}\lambda(x) = \delta_{m,n}, \qquad m, n \geqslant 0.$$

The distribution of the zeros of these polynomials is studied by means of a sequence of discrete measures $(\mu_n;\ n = 1, 2,\dots)$ defined by

$$\mu_n(\{x_{j,n}\}) = \frac{1}{n}, \qquad j = 1, 2,\dots, n,$$

$$\mu_n(A) = 0, \qquad A \text{ contains no zeros of } p_n(x)$$

where $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of $p_n(x)$. Let us decompose the spectral measure as $\lambda = \lambda_a + \lambda_s + \lambda_d$ where λ_a is absolutely continuous (with Radon–Nikodym derivative $w(x)$), λ_s is singular and λ_d is discrete having only mass points. It turns out that, as long as the behaviour of the absolutely continuous part is 'regular', the contribution of λ_s and λ_d has little effect on the zero behaviour. A well-known result of Erdös–Turán [2] states that if λ is concentrated on $[-1, 1]$ and if $w(x) > 0$ almost everywhere (in Lebesgue sense) on $[-1, 1]$ then μ_n will converge

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weakly to the measure μ given by

$$\mu(A)=\frac{1}{\pi}\int_A \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad A\subset[-1,1]$$

(this is the so-called arcsin law). This limit is independent of the spectral measure λ from as soon as the conditions above are fulfilled. As a consequence we have that under these conditions as $n\to\infty$

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu_n(x)=\frac{1}{n}\sum_{j=1}^{n} f(x_{j,n}) \;\to\; \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t$$

whenever f is a bounded measurable function on $[-1,1]$ so that the points of discontinuity form a set of Lebesgue measure zero.

Our aim is to find asymptotic results on

$$a(n)\left[\frac{1}{n}\sum_{j=1}^{n} f(x_{j,n})-\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t\right]$$

where $a(n)$ is an increasing sequence of real numbers. In this paper we will frequently use the Stieltjes transform of a (not necessarily positive) measure, defined as

$$S(\gamma;\,x)=\int\frac{\mathrm{d}\gamma(t)}{x-t}, \quad x\in\mathbb{C}\setminus\mathbb{R}.$$

An operation, closely connected to this transform, is the Stieltjes convolution of two measures, which we will define as a Schwarz distribution given by

$$\int f(x)\,\mathrm{d}\mu\,\Delta\lambda(x)=\iint\frac{f(x)-f(t)}{x-t}\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(t)$$

where f is an infinitely differentiable function with compact support. An important property of this operation is

$$S(\mu\Delta\lambda;\,x)=S(\mu;\,x)S(\lambda;\,x).$$

2. Results for the regular case

In our paper [7] we have supposed that the spectral measure λ is concentrated on $[-1,1]$ and that the Szegö condition is fulfilled:

$$\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}\,\mathrm{d}x>-\infty. \tag{2.1}$$

By means of the asymptotic relation

$$\frac{p_n(x)}{\left(x+\sqrt{x^2-1}\right)^n} \;\to\; \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\sqrt{x^2-1}}{2\pi}\int_{-1}^{1}\frac{\log\left[w(t)\sqrt{1-t^2}\right]}{\sqrt{1-t^2}}\,\frac{\mathrm{d}t}{x-t}\right\}$$

we were able to prove the following result:

Theorem 1 (Van Assche–Teugels [7]). *If $w(x)$ satisfies the Szegö condition* (2.1) *and if* $\log w(x)$ *is of bounded variation on every closed interval in* $(-1, 1)$, *then put*

$$d\gamma(x) = \frac{1}{2\pi}\sqrt{1-x^2}\, d\log\left[w(x)\sqrt{1-x^2}\right].$$

If γ defines a real measure on $[-1, 1]$ then as $n \to \infty$

(i) $\quad S(n[\mu_n - \mu]; x) \;\to\; S(\mu\Delta\gamma; x)$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$,

(ii) $$\sum_{j=1}^{n} f(x_{j,n}) - \frac{n}{\pi}\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt$$

$$\to \frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\sqrt{1-y^2}}{\sqrt{1-t^2}} \cdot \frac{f(y)-f(t)}{y-t}\, dy\, d\log\left[w(t)\sqrt{1-t^2}\right]$$

whenever f is analytic in some open set containing $[-1, 1]$.

For Jacobi polynomials, $w(x) = (1-x)^\alpha(1+x)^\beta (x \in [-1, 1], \alpha, \beta > -1)$ we found that $\mu\Delta\lambda$ is a measure given by

$$\mu\Delta\gamma(-\infty, x] = \frac{\alpha+\beta+1}{2}\mu(-\infty, x] - \frac{2\alpha+1}{4}U(x-1) - \frac{2\beta+1}{4}U(x+1)$$

where U is the Heaviside function

$$U(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geqslant 0. \end{cases}$$

The condition on f in (ii) can then be relaxed to f being continuous with a derivative that is absolutely integrable on $[-1, 1]$. Therefore the appropriate sequence for the rate in the arcsin law is $a(n) = n$.

Let us sketch the proof for a slightly more general case where the spectral measure λ, apart from the Szegö condition (2.1), also has a finite number of mass points outside $[-1, 1]$. Orthogonal polynomials always satisfy a recurrence formula of the type

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots$$

with $p_0 = 1$, $p_{-1} = 0$ and $a_n > 0$. If we suppose that

$$\sum_{n=1}^{\infty} n\left\{|1 - 4a_n^2| + 2|b_n|\right\} < \infty, \tag{2.2}$$

then the spectral measure λ will be of the form $\lambda = \lambda_a + \lambda_d$ where λ_a is an absolutely continuous measure that belongs to the Szegö class and λ_d is discrete with a finite number of mass points outside $[-1, 1]$. Combining results of Case and Geronimo [3] and Nevai ([5], Chapter 7) we find that as $n \to \infty$,

$$\frac{p_n(x)}{\left(x+\sqrt{x^2-1}\right)^n} \to \frac{R(x)}{\sqrt{2\pi}}\exp\left\{-\frac{\sqrt{x^2-1}}{2\pi}\int_{-1}^{1}\frac{\log\left[w(t)\sqrt{1-t^2}\right]}{\sqrt{1-t^2}}\,\frac{dt}{x-t}\right\}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$, where

$$R(x)=\prod_{z_i>0}(z_i-z)\prod_{z_j<0}(z-z_j)\Big/\prod_{k=1}^{N}(1-zz_k);$$

$$z=x-\sqrt{x^2-1},\quad z_i=x_i-\sqrt{x_i^2-1},$$

and $\{x_i;\ i=1,\dots,N\}$ are the points outside $[-1,1]$ where λ_d has positive mass. By taking derivatives in this asymptotic expression we obtain after calculations similar to those in [7]

$$\frac{p_n'(x)}{p_n(x)}-\frac{n}{\sqrt{x^2-1}} \;\to\; \frac{R'(x)}{R(x)}+\frac{1}{\sqrt{x^2-1}}\frac{1}{2\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{x-t}\,\mathrm{d}\log\left[w(t)\sqrt{1-t^2}\right]$$

and this holds uniformly on compact subsets of $\mathbb{C}\setminus([-1,1]\cup\{x_i\})$. We have to avoid the mass points x_i since $p_n(x_i)$ will tend to zero so that the left hand side is not bounded. The term on the left hand side is easily recognized as the Stieltjes transform on $n(\mu_n-\mu)$. The second expression on the right hand side is the product of two Stieltjes transforms and therefore the Stieltjes transform of a Stieltjes convolution $\mu\Delta\gamma$ where

$$\mathrm{d}\gamma(t)=\frac{1}{2\pi}\sqrt{1-t^2}\,\mathrm{d}\log\left[w(t)\sqrt{1-t^2}\right]. \tag{2.3}$$

Some elementary algebra leads to

$$\frac{R'(x)}{R(x)}=\frac{1}{\sqrt{x^2-1}}\sum_{j=1}^{N}\frac{\sqrt{x_i^2-1}}{x-x_j}=\sum_{j=1}^{N}\sqrt{x_j^2-1}\,S(\mu\Delta\gamma_j;\,x)$$

where $\gamma_j(-\infty,x]=U(x-x_j)$ is a Dirac measure at x_j. This leads to the following result:

Theorem 2. *Suppose that condition* (2.2) *is fulfilled and that* $\log w(t)$ *is of bounded variation on every closed interval in* $(-1,1)$. *If* γ *in* (2.3) *is a real measure on* $[-1,1]$ *then as* $n\to\infty$,

(i) $$S(n[\mu_n-\mu];\,x) \;\to\; \sum_{j=1}^{N}\sqrt{x_j^2-1}\,S(\mu\Delta\gamma_j;\,x)+S(\mu\Delta\gamma;\,x)$$

uniformly on compact subsets of $\mathbb{C}\setminus([-1,1]\cup\{x_i;\ i=1,\dots,N\})$ *where* x_i $(i=1,\dots,N)$ *are the mass points of the spectral measure* λ *outside* $[-1,1]$. γ_j *is a Dirac measure at* x_j *and* γ *is given in* (2.3),

(ii) $$\sum_{j=1}^{n}f(x_{j,n})-\frac{n}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t$$

$$\to\;\sum_{j=1}^{N}\sqrt{x_j^2-1}\,\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)-f(x_j)}{t-x_j}\,\frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

$$+\,\frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{f(t)-f(y)}{t-y}\,\frac{\sqrt{1-y^2}}{\sqrt{1-t^2}}\,\mathrm{d}y\,\mathrm{d}\log\left[w(t)\sqrt{1-t^2}\right]$$

whenever f *is analytic in some open set containing* $[a,b]$ *with* $a=\min(\min_j x_j,-1)$ *and* $b=\max(\max_j x_j,1)$.

The second part of this theorem goes as follows: take a closed contour Γ that encircles the interval $[a, b]$ and that is contained in the open set mentioned. Define the signed measure $\kappa_n = n(\mu_n - \mu)$, then by the theorems of Cauchy and Fubini

$$\int_a^b f(t)\,\mathrm{d}\kappa_n(t) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(z)\int_a^b \frac{\mathrm{d}\kappa_n(t)}{z-t}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(z) S(n[\mu_n - \mu];\, z)\,\mathrm{d}z.$$

Now let n tend to infinity (which causes no problems since Γ is compact) and use the first part of the theorem to obtain the result.

3. Results for Pollaczek polynomials

For the cases considered in Section 2 we found that the sequence $a(n) = n$ gave the appropriate rate. We will now investigate what happens if the Szegö condition (2.1) is violated. The Pollaczek polynomials are well known to have a spectral measure that doesn't satisfy (2.1) so that we get a good idea what happens by studying these polynomials. They are orthogonal with respect to an absolutely continuous measure with weight function

$$w(x) = 2\exp\left\{\frac{\operatorname{Arccos} x}{\sqrt{1-x^2}}(ax+b)\right\}\bigg/\left[1+\exp\left\{\frac{\pi}{\sqrt{1-x^2}}(ax+b)\right\}\right],$$
$$x \in [-1, 1], \quad a > |b|. \tag{3.1}$$

It is easy to see that $w(x) > 0$ on $(-1, 1)$ so that the Erdös–Turán result is still valid. The asymptotics for the Pollaczek polynomials are

$$\left(p_n(x)/\left(x+\sqrt{x^2-1}\right)^n\right) n^{-(ax+b)/2\sqrt{x^2-1}}$$
$$\to \frac{\left\{2\sqrt{x^2-1}\left(x-\sqrt{x^2-1}\right)\right\}^{-1/2+(ax+b)/2\sqrt{x^2-1}}}{\Gamma\left(1/2+(ax+b)/2\sqrt{x^2-1}\right)}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$ [6, appendix]. We proceed as in the previous section and take derivatives of this relation to obtain

$$\frac{n}{\log n}\left\{\frac{1}{n}\frac{p_n'(x)}{p_n(x)} - \frac{1}{\sqrt{x^2-1}}\right\} \to -\frac{1}{2}\frac{a+bx}{(x^2-1)\sqrt{x^2-1}}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$. The limit is equal to

$$-\frac{1}{4}\frac{1}{\sqrt{x^2-1}}\left(\frac{b-a}{x+1} + \frac{b+a}{x-1}\right)$$

which is easy to recognize as the Stieltjes transform of a sum of two Stieltjes convolutions. This leads to

Theorem 3. *Let $w(x)$ be the Pollaczek weight* (3.1), *then as $n \to \infty$,*

(i) $$S\left(\frac{n}{\log n}[\mu_n - \mu];\, x\right) \to -\frac{1}{4}\frac{1}{\sqrt{x^2-1}}\left\{\frac{b-a}{x+1} + \frac{b+a}{x-1}\right\}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$.

(ii)
$$\frac{1}{\log n}\sum_{j=1}^{n} f(x_{j,n}) - \frac{n}{\log n}\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t$$
$$\to -\frac{1}{4\pi}\left\{(b-a)\int_{-1}^{1}\frac{f(t)-f(-1)}{t+1}\frac{\mathrm{d}t}{\sqrt{1-t^2}} + (b+a)\int_{-1}^{1}\frac{f(t)-f(1)}{t-1}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\right\}$$

whenever f is analytic is some open set containing $[-1, 1]$.

The appropriate scaling now is $a(n) = n/\log n$, which is increasing more slowly than n. The behaviour of the Pollaczek weight at ± 1 is responsable for the fact that the Szegö condition is not satisfied, this is very apparent from Theorem 3 since the limit in (i) depends heavily on the points ± 1. Notice that for $a = b = 0$ we obtain a better rate, which is to be expected because these correspond to Legendre polynomials.

4. Results for Hermite polynomials

The result of Erdös–Turán is not valid when we consider polynomials with a spectral measure that is concentrated on an infinite interval, such as the Hermite polynomials, which are orthogonal on $(-\infty, \infty)$. The zeros of such polynomials spread out over the whole interval and are, in some cases, dense on $(-\infty, \infty)$. Therefore we should scale the zeros of such polynomials. For the Hermite polynomial of degree n it is known that the largest zero behaves like $\sqrt{2n}$ as $n \to \infty$ [6], which makes it natural to consider the normalized zeros $y_{j,n} = x_{j,n}/\sqrt{2n}$, which are the zeros of $p_n(\sqrt{2n}\,x\mathrm{t})$. Let us now define the contracted zero distribution by means of the following discrete measures:

$$\nu_n(\{y_{j,n}\}) = 1/n, \quad j = 1, 2, \ldots, n,$$
$$\nu_n(A) = 0, \qquad \text{if } A \text{ contains no } y_{j,n}.$$

It turns out that ν_n converges weakly to a measure ν on $[-1, 1]$ defined by

$$\nu(A) = \frac{2}{\pi}\int_A \sqrt{1-t^2}\,\mathrm{d}t, \quad A \subset [-1, 1].$$

(this is the so-called semi circle law). A consequence of this will be that

$$\frac{1}{n}\sum_{j=1}^{n} f\left(x_{j,n}/\sqrt{2n}\right) \to \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}\,\mathrm{d}t$$

for every bounded measurable function on $[-1, 1]$ having a set of discontinuity points of Lebesgue measure zero. We will prove this (known) result and give the exact rate of (4.1). There is a simple connection between the generalized Hermite polynomials $H_n^{(\alpha)}(x)$, which are orthogonal on $(-\infty, \infty)$ with respect to the weight function

$$w(x) = |x|^{2\alpha}\,\mathrm{e}^{-x^2}, \quad \alpha > -\tfrac{1}{2}. \tag{4.1}$$

and the Laguerre polynomials, namely [1, p. 156]

$$H_{2n}^{(\alpha)}(x) = (-1)^n 2^{2n} n! L_n^{(\alpha-1/2)}(x^2),$$
$$H_{2n+1}^{(\alpha)}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\alpha+1/2)}(x^2).$$

The following asymptotic behaviour for the Laguerre polynomials is known as the Plancherel–Rotach type asymptotic formula outside the oscillatory region [6, Theorem 8.22.8 (b)]

$$\mathrm{e}^{-x/2} L_n^{(\alpha)}(x) = \tfrac{1}{2}(-1)^n (\pi \sinh \phi)^{-1/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4}$$
$$\times \exp\{(n + \tfrac{1}{2}(\alpha+1))(2\phi - \sinh 2\phi)\}\{1 + \mathrm{O}(1/n)\}$$

where $x = (4n + 2\alpha + 2)\cosh^2\phi$ ($\varepsilon \leqslant \phi \leqslant \omega$) and the O-term holds uniformly. Let us take n even, then straightforward algebra leads to an asymptotic expression for the generalized Hermite polynomials: for $x > 1$

$$\frac{\sqrt{n}\, 2^{-n} H_n^{(\alpha)}(\sqrt{2n+2\alpha+1}\, x)}{(\frac{1}{2}n)!\left(x + \sqrt{x^2-1}\right)^n \left\{\exp\left(x\left(x - \sqrt{x^2-1}\right)\right)\right\}^n}$$
$$\to \frac{2^{-\alpha}}{\sqrt{2\pi}} \frac{\left\{\exp\left(x\left(x - \sqrt{x^2-1}\right)\right)\right\}^{\alpha+1/2}\left(x + \sqrt{x^2-1}\right)^{\alpha+1/2}}{x^\alpha (x^2-1)^{1/4}}$$

A similar result has also been proven in [4] using a quite different method. We need that this result holds uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$, in order to prove this we use the method of Theorem 7.1.1 in Szegö's book [6, p. 160] to obtain (n is still even)

$$2^{2n}\left(\frac{n}{2}\right)! \Gamma\left(\frac{n+2\alpha+1}{2}\right)$$
$$= \int_{-\infty}^{\infty} \left|H_n^{(\alpha)}(x)\right|^2 |x|^{2\alpha} \mathrm{e}^{-x^2}\, \mathrm{d}x$$
$$\geqslant (2n+2\alpha+1)^{\alpha+1/2} \int_{-1}^{1} \left|H_n^{(\alpha)}(\sqrt{2n+2\alpha+1}\, x)\right|^2 |x|^{2\alpha} \mathrm{e}^{-(2n+2\alpha+1)x^2}\, \mathrm{d}x$$
$$\geqslant (2n+2\alpha+1)^{\alpha+1/2} \pi\left(1 - |z|^2\right)\left|H_n^{(\alpha)}(\sqrt{2n+2\alpha+1}\, x) z^n\right|^2 |D_n(z)|^2,$$

where $z = x - \sqrt{x^2-1}$ and

$$D_n(z) = \exp\left\{-\frac{2n+2\alpha+1}{4\pi}\int_{-\pi}^{\pi} \cos^2 t \frac{1 + z\,\mathrm{e}^{-\mathrm{i}t}}{1 - z\,\mathrm{e}^{-\mathrm{i}t}}\, \mathrm{d}t\right\}.$$
$$\times \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(|\cos t|^{2\alpha} |\sin t|\right) \frac{1 + z\,\mathrm{e}^{-\mathrm{i}t}}{1 - z\,\mathrm{e}^{-\mathrm{i}t}}\, \mathrm{d}t\right\}$$
$$= \left\{\exp\left(x\left(x - \sqrt{x^2-1}\right)\right)\right\}^{-(n+\alpha+1/2)} \left\{x\left(x - \sqrt{x^2-1}\right)\right\}^{\alpha}$$
$$\times \left(x + \sqrt{x^2-1}\right)^{-1/2} (x^2-1)^{1/4},$$

which gives the bound

$$\left| \frac{(n+\alpha+1/2)^{\alpha/2+1/4} 2^{-n} H_n^{(\alpha)}(\sqrt{2n+2\alpha+1}\, x)}{\sqrt{\left(\frac{n}{2}\right)! \Gamma\left(\frac{n+2\alpha+1}{2}\right)} \left(x+\sqrt{x^2-1}\right)^n \left\{\exp\left(x\left(x-\sqrt{x^2-1}\right)\right)\right\}^n} \right|$$

$$\leqslant \frac{2^{-\alpha/2-1/4}}{\sqrt{\pi}\sqrt{1-|z|^2}} \left|\exp\left(x\left(x-\sqrt{x^2-1}\right)\right)\right|^{\alpha+1/2} \left|\frac{x+\sqrt{x^2-1}}{x}\right|^{\alpha} \left|\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right|^{1/2},$$

and the right hand side can be uniformly bounded if x is in an compact subset of $\mathbb{C} \setminus [-1, 1]$. Now we know that the left hand side converges on a subset of $\mathbb{C} \setminus [-1, 1]$ with an accumulation point, from which the uniform convergence on compact subsets follows by Vitali's theorem. A similar reasoning gives the same result for odd values of n. From now on we set $p_n(x) = H_n^{(\alpha)}(x)$, and because of the uniform convergence we may now differentiate the asymptotic relation to obtain

$$\sqrt{1+2\alpha+2n}\, \frac{p_n'(\sqrt{1+2\alpha+2n}\, x)}{p_n(\sqrt{1+2\alpha+2n}\, x)} - \frac{2n}{x+\sqrt{x^2-1}}$$

$$\to \frac{1+2\alpha}{2} \frac{2}{x+\sqrt{x^2-1}} - \frac{\alpha}{x} - \frac{1}{2} \frac{x}{x^2-1}.$$

Now notice that

$$S(\nu;\, x) = 2/\left(x+\sqrt{x^2-1}\right)$$

so that we find the following

Theorem 4. *For the zeros of the generalized Hermite polynomials with weight* (4.1)

(i) $$S(n[\nu_n - \nu];\, x) \to \frac{1+2\alpha}{2} \frac{2}{x+\sqrt{x^2-1}} - \frac{\alpha}{x} - \frac{1}{4}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

for $x \in \mathbb{C} \setminus [-1, 1]$, *where* ν_n *is the contracted zero distribution at the points* $y_{j,n} = x_{j,n}/\sqrt{2n+2\alpha+1}$,

(ii) $$\sum_{j=1}^{n} f\left(\frac{x_{j,n}}{\sqrt{2n+2\alpha+1}}\right) - \frac{2n}{\pi} \int_{-1}^{1} f(t)\sqrt{1-t^2}\, dt$$

$$\to \frac{1+2\alpha}{2} \frac{2}{\pi} \int_{-1}^{1} f(t)\sqrt{1-t^2}\, dt - \alpha f(0) - \frac{1}{4}\{f(1) + f(-1)\}$$

whenever f is analytic in some open set containing $[-1, 1]$.

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